RELATIVITY AND COSMOLOGY I

Solutions to Problem Set 4 — Fall 2023

1. The Levi-Civita Tensor

In Problem Set 2 you showed that, for a generic rank 2 matrix,

$$\tilde{\epsilon}_{\mu_1\cdots\mu_n} M^{\mu_1}{}_{\mu'_1} \cdots M^{\mu_n}{}_{\mu'_n} = \det(M)\tilde{\epsilon}_{\mu'_1\cdots\mu'_n} \,. \tag{1}$$

(a) By choosing to consider the matrix of a coordinate transformation $M^{\mu'}{}_{\mu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}}$, we get

$$\tilde{\epsilon}_{\mu'_1\cdots\mu'_n} = \det\left(\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right) \frac{\partial x^{\mu_1}}{\partial x^{\mu'_1}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu'_n}} \tilde{\epsilon}_{\mu_1\cdots\mu_n} \,. \tag{2}$$

(b) Using the transformation law for metric

$$\begin{aligned} g(x') = \det(g'_{\mu'\nu'}) &= \det\left(\frac{\partial x^{\mu}}{\partial x^{\mu'}} g_{\mu\nu} \frac{\partial x^{\nu}}{\partial x^{\nu'}}\right) = \\ &= \det\left(\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right) \det(g_{\mu\nu}) \left(\frac{\partial x^{\nu}}{\partial x^{\nu'}}\right) = \left(\det\left(\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right)\right)^{-2} g(x) \end{aligned} \tag{3}$$

where properties $\det(AB) = \det(A)\det(B)$ and $\det(A^{-1}) = \frac{1}{\det(A)}$ were used.

(c) Via the transformation rules established in previous parts,

$$\begin{aligned} \epsilon_{\mu_1\ldots\mu_n} = \sqrt{|g|}\tilde{\epsilon}_{\mu_1\ldots\mu_n} &= \frac{\sqrt{|g'|}}{\left|\det\left(\frac{\partial x}{\partial x'}\right)\right|} \det\left(\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right) \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu'_n}}{\partial x^{\mu_n}} \tilde{\epsilon}_{\mu'_1\ldots\mu'_n} = \\ &= \pm\sqrt{|g'|} \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu'_n}}{\partial x^{\mu_n}} \tilde{\epsilon}_{\mu'_1\ldots\mu'_n} = \pm \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu'_n}}{\partial x^{\mu_n}} \epsilon_{\mu'_1\ldots\mu'_n} \end{aligned} \tag{4}$$

where the final sign depends on sign of $\det\left(\frac{\partial x}{\partial x'}\right)$.

(d) We have that

$$\begin{aligned} \epsilon &= \frac{1}{n!}\epsilon_{\mu_1\cdots\mu_n} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} \\ &= \frac{1}{n!}\sqrt{|g|}\tilde{\epsilon}_{\mu_1\cdots\mu_n} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} \\ &= \sqrt{|g|}\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1} \\ &= \sqrt{|g|}d^n x \,. \end{aligned} \tag{5}$$

Why do we say that $\tilde{\epsilon}$ transform ? We defined $\tilde{\epsilon}$ as a symbol taking values ± 1 and 0. Why should it transform ? Tensor components transform because they are maps from copies of the tangent and cotangent spaces to the real numbers, and bases of those vector spaces

depend on the choice of chart (=coordinates) on the manifold. Therefore people using different coordinates will use different tensor components to describe the same physics. But $\tilde{\epsilon}$ is not a tensor field, nor is ϵ. So why do we say it transforms ? It is not because we define something with indices that it must change under a coordinates transformation.
The point is that the index notation is convenient to write scalar quantities - things that are invariant under a change of coordinates. For example $V^\mu(x)W_\mu(x)$ will be mapped to itself under a coordinate transformation.
Now we could write $\epsilon_{\mu\nu\rho}T^{\mu\nu\rho}(x)$ for some tensor field T. What happens to this expression when we do a coordinates transformation ? Let's take the point of view where $\tilde{\epsilon}$ is just a symbol and therefore is invariant under a change of coordinates, even if it has indices. We get

$$\begin{aligned}\epsilon_{\mu\nu\rho}T^{\mu\nu\rho}(x) &= \tilde{\epsilon}_{\mu\nu\rho}\sqrt{g(x)}T^{\mu\nu\rho}(x) \\ &\to \tilde{\epsilon}_{\mu\nu\rho}\sqrt{g'(x')}T'^{\mu\nu\rho}(x') = \tilde{\epsilon}_{\mu\nu\rho}\det\left(\frac{\partial x}{\partial x'}\right)\sqrt{g(x)}\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}\frac{\partial x^\rho}{\partial x'^\gamma}T^{\alpha\beta\gamma}(x)\end{aligned} \tag{6}$$

Now we use the definition of the determinant of a matrix

$$\epsilon_{\mu\nu\rho}\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}\frac{\partial x^\rho}{\partial x'^\gamma} = \epsilon_{\alpha\beta\gamma}\det\left(\frac{\partial x'}{\partial x}\right), \tag{7}$$

so that the determinant of $\frac{\partial x'}{\partial x}$ and its inverse $\frac{\partial x}{\partial x'}$ cancel in the above expression and we're left with

$$\tilde{\epsilon}_{\mu\nu\rho}\sqrt{g'(x')}T'^{\mu\nu\rho}(x') = \tilde{\epsilon}_{\alpha\beta\gamma}\sqrt{g(x)}T^{\alpha\beta\gamma}(x), \tag{8}$$

showing that the expression indeed transforms as a scalar field. We would get the same answer by saying that $\epsilon_{\mu\nu\rho}$ is effectively a rank 3 covariant tensor field, and transform it as if it were, so this is why many people and textbooks say it. This is an abuse of language, you can say it to make calculations quicker, but you should keep in mind the physics and what the correct computation is.

2. Some Definitions and Questions

(a) With tensor transformation rules one has

$$W_{\mu'} = \left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right)W_\mu\,. \tag{9}$$

Therefore

$$\begin{aligned}\partial_{\mu'}W_{\nu'} &= \partial_{\mu'}\left(\left(\frac{\partial x^\nu}{\partial x^{\nu'}}\right)W_\nu\right) \\ &= \left(\frac{\partial x^\nu}{\partial x^{\nu'}}\right)\partial_{\mu'}W_\nu + \left(\frac{\partial^2 x^\nu}{\partial x^{\nu'}\partial x^{\mu'}}\right)W_\nu \\ &= \frac{\partial x^\nu}{\partial x^{\nu'}}\frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu W_\nu + \left(\frac{\partial^2 x^\nu}{\partial x^{\nu'}\partial x^{\mu'}}\right)W_\nu\end{aligned} \tag{10}$$

where the first term in the sum is the expected tensor transformation, and the second term in the sum is, in general, non-zero, making $\partial_\mu W_\nu$ **not** a tensor.

(b) Using the previous expression, one gets

$$\partial_{[\mu'} W_{\nu']} = \left(\frac{\partial x^\nu}{\partial x^{[\nu'}}\right)\left(\frac{\partial x^\mu}{\partial x^{\mu']}}\right)\partial_\mu W_\nu = \left(\frac{\partial x^\nu}{\partial x^{\nu'}}\right)\left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right)\partial_{[\mu} W_{\nu]} \tag{11}$$

where the offending term disappeared, since partial derivatives commute.

(c) Expanding the brackets and using the Leibniz rule gives

$$\begin{aligned}[V, W] &= V^\mu \partial_\mu (W^\nu \partial_\nu) - W^\mu \partial_\mu (V^\nu \partial_\nu) = \\ &= V^\mu \partial_\mu W^\nu \partial_\nu + V^\mu W^\nu \partial_\mu \partial_\nu - W^\mu \partial_\mu V^\nu \partial_\nu - W^\mu V^\nu \partial_\mu \partial_\nu = \\ &= (V^\mu \partial_\mu W^\nu - W^\mu \partial_\mu V^\nu)\partial_\nu \,.\end{aligned} \tag{12}$$

In components, we thus have that

$$[V, W]^\nu = V^\mu \partial_\mu W^\nu - W^\mu \partial_\mu V^\nu \,. \tag{13}$$

(d) We want to show that $(**A) \propto A$. From the definition,

$$\begin{aligned}(**A)_{\mu_1 \cdots \mu_p} &= \frac{1}{p!}\frac{1}{(n-p)!}\epsilon^{\sigma_1 \cdots \sigma_{n-p}}{}_{\mu_1 \cdots \mu_p}\epsilon^{\nu_1 \cdots \nu_p}{}_{\sigma_1 \cdots \sigma_{n-p}} A_{\nu_1 \cdots \nu_p} \\ &= \frac{1}{p!}\frac{1}{(n-p)!}(-1)^{p(n-p)}\epsilon^{\sigma_1 \cdots \sigma_{n-p}}{}_{\mu_1 \cdots \mu_p}\epsilon_{\sigma_1 \cdots \sigma_{n-p}}{}^{\nu_1 \cdots \nu_p} A_{\nu_1 \cdots \nu_p} \\ &= (-1)^{s+p(n-p)}\delta^{[\nu_1}_{[\mu_1} \cdots \delta^{\nu_p]}_{\mu_p]} A_{\nu_1 \cdots \nu_p} \\ &= (-1)^{s+p(n-p)} A_{\mu_1 \cdots \mu_p} \,,\end{aligned} \tag{14}$$

as we wanted to prove. We used the identity

$$\begin{aligned}\epsilon^{\mu_1 \ldots \mu_n} &= g^{\mu_1 \nu_1} g^{\mu_2 \nu_2} \cdots g^{\mu_n \nu_n} \epsilon_{\nu_1 \ldots \nu_n} \\ &= g^{\mu_1 \nu_1} g^{\mu_2 \nu_2} \cdots g^{\mu_n \nu_n} \tilde{\epsilon}_{\nu_1 \ldots \nu_n} \sqrt{g} \\ &= \frac{\sqrt{g}}{g}\tilde{\epsilon}^{\mu_1 \ldots \mu_n} = \frac{1}{\sqrt{g}}\tilde{\epsilon}^{\mu_1 \ldots \mu_n},\end{aligned} \tag{15}$$

so that factors of $\sqrt{g}$ cancel when contracting p upper indices of the ϵ tensor with p lower indices of another ϵ tensor.

3. The Dimensionality of Space

(a) One way of determining the dimensionality of these spaces, is to look at the determinant of the metric. If it is zero, it means some of the coordinates are linearly dependent on one another. In matrix form, the first line element can be written as

$$ds^2 = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \tag{16}$$

The first column is linearly dependent on the second and on the third: this matrix has rank 1. That means that the space is one-dimensional. We can figure that out also by realizing that

$$\begin{aligned} ds^2 &= \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - (\mathrm{d}x\mathrm{d}y + \mathrm{d}y\mathrm{d}x) + (\mathrm{d}x\mathrm{d}z + \mathrm{d}z\mathrm{d}x) - (\mathrm{d}y\mathrm{d}z + \mathrm{d}z\mathrm{d}y) \\ &= (\mathrm{d}x - \mathrm{d}y + \mathrm{d}z)^2 \\ &= \mathrm{d}u^2 \,, \end{aligned} \tag{17}$$

where we did a change of variables $u = x - y + z$.

(b) Here we realize that

$$\begin{aligned} ds^2 &= \mathrm{d}r^2 + r^2\Big(\mathrm{d}\theta_1^2 + \mathrm{d}\theta_2^2 + (\mathrm{d}\theta_1\mathrm{d}\theta_2 + \mathrm{d}\theta_2\mathrm{d}\theta_1) \\ &\quad + \Big(\sin^2\theta_1\cos^2\theta_2 + \sin^2\theta_2\cos^2\theta_1 + \frac{1}{2}\sin 2\theta_1 \sin 2\theta_2\Big)\mathrm{d}\phi^2\Big) \\ &= \mathrm{d}r^2 + r^2(\mathrm{d}(\theta_1+\theta_2))^2 + r^2\sin^2(\theta_1+\theta_2)\mathrm{d}\phi^2 \,. \end{aligned} \tag{18}$$

By redefining $\theta = \theta_1 + \theta_2$ we get

$$ds^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2 \,, \tag{19}$$

where we recognize the metric of $\mathbb{R}^3$ in spherical polar coordinates.

Notice that the metric is defined to be non-degenerate and positive definite, such that $g_{\mu\nu}V^\mu V^\nu > 0$ and $g_{\mu\nu}V^\mu V^\nu = 0$ if and only if $V^\mu = 0$. That means the rank of the metric tells you about the dimensionality of the manifold in which you are.

4. Electromagnetism with p-forms

(a) We start by computing

$$(*F)_{\alpha\beta} = \frac{1}{2}\epsilon^{\mu\nu}{}_{\alpha\beta}F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu} \,. \tag{20}$$

Then, acting with the exterior derivative, we get

$$(\mathrm{d}(*F))_{\gamma\alpha\beta} = \frac{3}{2}\partial_{[\gamma}\epsilon_{\alpha\beta]\mu\nu}F^{\mu\nu} \,. \tag{21}$$

At the same time,

$$(*J)_{\gamma\alpha\beta} = \epsilon^{\rho}{}_{\gamma\alpha\beta}J_\rho = \epsilon_{\rho\gamma\alpha\beta}J^\rho \,. \tag{22}$$

We thus want to show that

$$\frac{3}{2}\partial_{[\gamma}\epsilon_{\alpha\beta]\mu\nu}F^{\mu\nu} = -\epsilon_{\gamma\alpha\beta\rho}J^\rho \tag{23}$$

is equivalent to Maxwell's equation. For that purpose, let us act on both sides with $\epsilon^{\gamma\alpha\beta\tau}$. We get

$$\begin{aligned}
\frac{3}{2}\epsilon^{\gamma\alpha\beta\tau}\partial_{[\gamma}\epsilon_{\alpha\beta]\mu\nu}F^{\mu\nu} &= -\epsilon^{\gamma\alpha\beta\tau}\epsilon_{\gamma\alpha\beta\rho}J^{\rho} \\
\frac{3}{2}\epsilon^{\gamma\alpha\beta\tau}\epsilon_{\alpha\beta\mu\nu}\partial_{\gamma}F^{\mu\nu} &= 3!\delta^{\tau}_{\rho}J^{\rho} \\
\frac{3}{2}\epsilon^{\alpha\beta\gamma\tau}\epsilon_{\alpha\beta\mu\nu}\partial_{\gamma}F^{\mu\nu} &= 6J^{\tau} \\
-\frac{3}{2}2!2!\delta^{[\gamma}_{[\mu}\delta^{\tau]}_{\nu]}\partial_{\gamma}F^{\mu\nu} &= 6J^{\tau} \\
\partial_{\mu}F^{\tau\mu} &= J^{\tau}\,,
\end{aligned} \tag{24}$$

which contains two of Maxwell's equations. The other computation is easier

$$(\mathrm{d}F)_{\alpha\beta\gamma} = 3\partial_{[\alpha}F_{\beta\gamma]}\,. \tag{25}$$

We want to show that

$$\partial_{[\alpha}F_{\beta\gamma]} = 0 \tag{26}$$

reduces to Maxwell's equations. There is only one step to do: act with $\epsilon^{\rho\alpha\beta\gamma}$ to get

$$\begin{aligned}
\epsilon^{\rho\alpha\beta\gamma}\partial_{[\alpha}F_{\beta\gamma]} &= 0 \\
\epsilon^{\rho\alpha\beta\gamma}\partial_{\alpha}F_{\beta\gamma} &= 0
\end{aligned} \tag{27}$$

which is exactly the form in which we saw Maxwell's equations in Problem Set 2. In this language, Maxwell's equations are thus

$$\mathrm{d}*F = *J\,, \qquad \mathrm{d}F = 0\,. \tag{28}$$

(b) From $\mathrm{d}*F = *J$ we directly obtain

$$\mathrm{d}*J = \mathrm{d}^2*F = 0. \tag{29}$$

by using the property of the exterior derivative $\mathrm{d}^2 = 0$. To evaluate this equation in components we start by noticing that since J is a 1-form, then $(*J)$ is a 3 form with components

$$(*J)_{\mu\nu\rho} = \epsilon^{\alpha}{}_{\mu\nu\rho}J_{\alpha}. \tag{30}$$

Taking the exterior derivative we have

$$0 = (\mathrm{d}*J)_{\sigma\mu\nu\rho} = 4\partial_{[\sigma}(*J)_{\mu\nu\rho]} = 4\partial_{[\sigma}\epsilon^{\alpha}{}_{\mu\nu\rho]}J_{\alpha}. \tag{31}$$

The easiest way to proceed is to contract this expression with $\epsilon^{\mu\nu\rho\sigma}$ which gives

$$0 = 4\epsilon^{\mu\nu\rho\sigma}\partial_{[\sigma}\epsilon^{\alpha}{}_{\mu\nu\rho]}J_{\alpha} = 4\epsilon^{\mu\nu\rho\sigma}\partial_{\sigma}\epsilon^{\alpha}{}_{\mu\nu\rho}J_{\alpha} = 4(-1)(3!)\partial_{\sigma}\eta^{\alpha\sigma}J_{\alpha}, \tag{32}$$

which implies the continuity equation

$$0 = \partial_{\alpha}J^{\alpha} = \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J}. \tag{33}$$

(c) We are given that the only non-zero components of $(*F)$ are

$$(*F)_{\theta\phi} = q\sin\theta\,, \qquad (*F)_{\phi\theta} = -q\sin\theta\,. \tag{34}$$

If we act with another Hodge dual operator, we will obtain $**F = -F$. From the definition,

$$
\begin{aligned}
(**F)_{\alpha\beta} &= \frac{1}{2}\epsilon^{\mu\nu}{}_{\alpha\beta}(*F)_{\mu\nu} \\
&= \frac{1}{2}\left(\epsilon^{\theta\phi}{}_{\alpha\beta}(*F)_{\theta\phi} + \epsilon^{\phi\theta}{}_{\alpha\beta}(*F)_{\phi\theta}\right) \\
&= \frac{1}{2}\left(g^{\theta\mu}g^{\nu\phi}\epsilon_{\mu\nu\alpha\beta}q\sin\theta - g^{\phi\lambda}g^{\tau\theta}\epsilon_{\lambda\tau\alpha\beta}q\sin\theta\right) \\
&= \frac{1}{2}\left(g^{\theta\theta}g^{\phi\phi}\epsilon_{\theta\phi\alpha\beta}q\sin\theta - g^{\phi\phi}g^{\theta\theta}\epsilon_{\phi\theta\alpha\beta}q\sin\theta\right) \\
&= \frac{1}{2}\frac{1}{r^4\sin^2\theta}\sqrt{|g|}q\sin\theta\left(\tilde{\epsilon}_{\theta\phi\alpha\beta} - \tilde{\epsilon}_{\phi\theta\alpha\beta}\right) \\
&= \frac{1}{2}\frac{q}{r^2}\left(\tilde{\epsilon}_{\theta\phi\alpha\beta} - \tilde{\epsilon}_{\phi\theta\alpha\beta}\right) .
\end{aligned}
\tag{35}
$$

The only non-vanishing components, because of the full antisymmetricity of $\tilde{\epsilon}$, are

$$
(**F)_{tr} = \frac{q}{r^2}\,, \qquad (**F)_{rt} = -\frac{q}{r^2}\,, \tag{36}
$$

implying

$$
F_{tr} = -\frac{q}{r^2}\,, \qquad F_{rt} = \frac{q}{r^2}\,. \tag{37}
$$

(d) We can read off that all magnetic fields vanish, and the electric field is radial

$$
E_r = F^{tr} = \frac{q}{r^2}\,. \tag{38}
$$

This is the electric field generated by a point particle placed at the origin.

(e) The generalized Stokes' theorem reads

$$
\int_V \mathrm{d}\omega = \int_{\partial V}\omega\,, \tag{39}
$$

where ω is an $n-1$ form and V is an n-dimensional manifold with boundary ∂V. In our case, we have

$$
\int_V \mathrm{d}*F = \int_{\partial V} *F = \int_{\partial V} q\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\phi = q\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\theta\sin\theta = 4\pi q\,, \tag{40}
$$

such that we retrieve Gauss' law.